JOURNAL OF FUNCTIONAL ANALYSIS **50**, 317–328 (1983)

Normal State Spaces of Jordan and von Neumann Algebras

BRUNO IOCHUM

Université de Provence and Centre de Physique Théorique, CNRS, Marseille, Cédex 9, France

AND

FREDERIC W. SHULTZ

Department of Mathematics, Wellesley College, Wellesley, Massachusetts

Communicated by the Editors

Received December 18, 1981

Normal state spaces of Jordan and von Neumann algebras are characterized among convex sets. The normal state spaces of JBW-algebras are precisely those which are spectral and elliptic; along these the normal state spaces of von Neumann algebras are distinguished by the global 3-ball property.

NORMAL STATE SPACES OF JORDAN AND VON NEUMANN ALGEBRAS

Recently, geometric characterizations have been given for the state spaces of the normed Jordan algebras known as JB-algebras [5] and of state spaces for C^*-algebras [6]. These characterizations are expressed in terms of conditions which involve the pure states in an essential way, and this cannot readily by carried over to the context of normal state spaces of Jordan and von Neumann algebras, where there may not be any extreme points. Our purpose here is to give geometric axioms which characterize these normal state spaces.

We begin with the context of a spectral convex set: see [2, 3] for background. We next add the requirement of "ellipticity"—that certain cross sections are elliptic. This notion is due to Alfsen [4], who showed it was closely related to Connes' "facial homogeneity" axiom [12] transported to the context of spectral convex sets. In Section 1 below we show that a convex set is affinely isomorphic to the normal state space of a JBW-algebra iff it is spectral and elliptic. (This answers a question raised in [4].)

Then we introduce another axiom: the "global 3-ball property." This

0022-1236/83 $3.00

involves the facial structure, and is a reformulation of the 3-ball property of [6] in a context where there may be no extreme points. This new axiom distinguishes normal state spaces of von Neumann algebras among those of JBW-algebras. Thus we obtain a characterization of normal state spaces of von Neumann algebras: they are precisely the spectral, elliptic convex set with the global 3-ball property.

We remark that there is a related set of results which characterize the "natural" cones associated with JBW and von Neumann algebras [8–12]. Finally, we could like to express our appreciation to Erik Alfsen for permission to use his result connecting "facial homogeneity" (suitably stated) and ellipticity. (This previously appeared only in the preprint [4].)

1. Ellipticity and Normal State Spaces of JBW-Algebras

The notion of ellipticity was introduced in [4, Sect. 5]. Since the final version of that paper omitted this discussion, for the convenience of the reader we have reproduced the facts we will need (Lemmas 1.1 and 1.4). We assume (throughout this paper) familiarity with basic facts about spectral theory for convex sets as described in [2, 3] and relevant background on JB and JBW-algebras and their state spaces [5, 7, 15].

We recall from [4] that a convex set K is spectral if K can be embedded in a linear space V in such a way that (V, K) becomes a base-norm space in spectral duality (cf. [2]) with the order unit space (A, e), where $A = A^b(K) \approx V^*$ and $e = 1$ is the distinguished order unit. The above definition is more general than that of [2] which is confined to compact convex sets. Note, however, that the new definition will agree with the old one when K is compact since every compact convex set can be embedded in a locally convex space in such a way that (V, K) becomes a base-norm space (the "regular embedding" of [1, Chap. II, Sect. 2]. Note that if P is a P-projection on A with quasi-complement P', then $F = K \cap \operatorname{Im} P^*$ and $F^\# = K \cap \operatorname{Im} P'^*$ are quasi-complementary projective faces of K and there exists a unique affine retraction ψ_F of K onto $\operatorname{co}(F \cup F^\#)$, namely, $\psi_F = (P + P')^*|_K$ (cf. [2, Theorem 3.8]). Following [2, Theorem 10.6], we say that a compact convex set K is strongly spectral if it is spectral and if $A(K)$ is closed under functional calculus by continuous functions (in particular $A(K)$ is closed under the squaring map). (See [2] for a more geometric definition.)

Lemma 1.1 [4, Lemma 5.1]. *Let K be a spectral convex set. Let P be a P-projection with quasi-complement P' on A and ρ an element of K such that $P^*\rho \neq 0$ and $P^*\rho \neq \rho$. Moreover, let $\sigma = \|P^*\rho\|^{-1}P^*\rho$, $\tau = \|P'^*\rho\|^{-1}P'^*\rho$,* *and let* $\psi_F = (P+P')^*|_K$ *be the unique affine retraction of* K *onto* $\mathrm{co}(F \cup F^\#)$. *If* ρ *is not contained in the line segment* $[\sigma, \tau]$, *then as* t *goes from* $-\infty$ *to* $+\infty$ *the point*

$$\rho_t = \frac{\exp t(P-P')^*\rho}{\langle e, \exp t(P-P')^*\rho\rangle}$$

will describe one half of the (unique) ellipse $E_p(\rho)$ *through* ρ *which has* $[\sigma, \tau]$ *as one diameter and has the conjugate diameter in the direction of the vector* $\rho - \psi_F(\rho)$. *If* $\rho \in [\sigma, \tau]$, *then as* t *goes from* $-\infty$ *to* $+\infty$ *the point* ρ_t *will describe the "degenerate ellipse"* $E_P(\rho) = [\sigma, \tau]$.

Proof. Note that the unrenormalized trajectory

$$\exp t(P-P')^*\rho = e^t P^*\rho + e^{-t}P'^*\rho + (\mathbb{1} - P - P')^*\rho$$

is one half of a hyperbola. When this hyperbola is projected on $e^{-1}(1)$, the image is a bounded conic section missing only the points σ and τ; this is precisely the ellipse passing through σ, τ, and ρ.

For an explicit coordinatization of this ellipse, let $\alpha = \|P^*\rho\|$, so $1-\alpha = \|P'^*\rho\|$, and write

$$\rho_t = x(t)\sigma + y(t)z + z(t)(\mathbb{1} - P - P')^*\rho.$$

One verifies that $x(t) + y(t) = 1$ and $z(t)^2 = \alpha^{-1}(1-\alpha)^{-1}x(t)\,y(t)$ and that as t varies from $-\infty$ to $+\infty$, then $x(t)$ varies from 0 to 1; setting $x'(t) = x(t) - \frac{1}{2}$ gives

$$\rho_t = \tfrac{1}{2}(\sigma + \tau) + x'(t)(\sigma - \tau) + z(t)(\rho - \psi_F\rho),$$

where $\frac{1}{2} < x'(t) < \frac{1}{2}$ and $z(t) = [\alpha^{-1}(1-\alpha)^{-1}(\frac{1}{4} - x'(t)^2)]^{1/2}$; this gives the semi-ellipse described above.

Definition 1.2. A spectral convex set K is *elliptic* if the ellipses $E_P(\rho)$ are contained in K for all P-projections P and all ρ in K.

Remark 1.3. (i) By Lemma 1.1, the "top" half of the ellipses $E_P(\rho)$ are contained in K for all ρ in K iff

$$\exp\,(t(P-P') \geqslant 0 \qquad \text{for all} \quad t \in \mathbb{R}. \tag{1}$$

(ii) Ellipticity can also be characterized in terms of the geometry of the fibers $\psi_F^{-1}([\sigma, \tau])$, where F is any projective face of K and $\sigma \in F$, $\tau \in F^\#$. Such a fiber is said to have elliptic cross sections if every plane M (i.e., two-dimensional affine subspace) through $[\sigma, \tau]$ meets the fiber in an elliptic disk (i.e., the convex hull of an ellipse). It has been proved in [4, p. 59] that a spectral convex set is elliptic iff the fibers have elliptic cross sections. The proof is not difficult and we will only recall the idea: assuming that K has elliptic cross sections, we consider the projective face F associated to the P-projection P. Let $\rho \in K$ and let σ, τ be as in Lemma 1.1. If E is the intersection of $\psi_F^{-1}([\sigma, \tau])$ with the plane generated by σ, τ, and ρ, then E is an elliptic disk and has ψ_F as affine retraction onto $[\sigma, \tau]$. Therefore $\psi_F(\rho) - \rho$ points in the direction of the diameter of E which is conjugate to $[\sigma, \tau]$; thus the diameter of E is colinear with a diameter of $\mathrm{co}(E_p(\rho))$. Since $\rho \in \partial_e \mathrm{co}(E_p(\rho)) \cap K$ we conclude that $E_p(\rho) \subset E \subset K$. The converse is immediate.

Note that K is the union of the fibers $\psi_F^{-1}([\sigma, \tau])$, where F is a projective face, $\sigma \in F$ and $\tau \in F^{\#}$. These fibers will be either disjoint or they will meet at an "end point" $\sigma \in F$ or $\tau \in F^{\#}$. If F is a split face, then $\psi_F^{-1}(([\sigma, \cdot]) = [\sigma, \tau]$.

LEMMA 1.4. [4, Proposition 5.3]. *The normal state space of a* JBW-*algebra* A *is elliptic.*

Proof. We first establish (1). Let P be a P-projection on A, and let u be the idempotent in A such that $Pa = \{u, a, u\}$ for $a \in A$ (cf. [3, Proposition 3.1]). From the definition of the Jordan triple product, for $a \in A$, $\lambda \in \mathbb{R}^+$, and $u' = e - u$.

$$(\lambda P + \mathbb{1} - P - P' + \lambda^{-1}P')a = \{\lambda^{1/2}u + \lambda^{-1/2}u', a, \lambda^{1/2}u + \lambda^{-1/2}u'\}.$$

By [7, Proposition 2.7], the maps $a \to \{b, a, b\}$ are positive for all $b \in A$. Therefore

$$\lambda P + \mathbb{1} - P - P' + \lambda^{-1}P' \geqslant 0 \qquad \text{for all} \quad \lambda \in \mathbb{R}^+.$$

If λ is replaced by e^t, the expression on the left is precisely $\exp t(P - P')$, which by Lemma 1.1 shows the upper half of the ellipse $E_p(\rho)$ is contained in K for all ρ in K. But now symmetry of K with respect to $\mathrm{co}(F \cup F^{\#})$ as described in [5, Proposition 3.14], and in Section 2, implies that all of $E_p(\rho)$ is in K.

THEOREM 1.5. *A convex set K is affinely isomorphic to the normal state space of a* JBW-*algebra iff K is spectral and elliptic.*

Proof. The normal state spaces of a JBW-algebra is spectral by [3, Proposition 3.1] and elliptic by Lemma 1.4.

Conversely, assume K is a convex set which is spectral and elliptic. We may assume that K is the base of a base norm space (V, K), and that (V, K) and $(A = V^*, e)$ are in spectral duality, where e is the canonical order unit.

We will show A can be given the structure of a JBW-algebra, and that K can be identified with its normal state space.

Let P and Q be any P-projections on A. We will first establish the identity

$$Q'(P-P')Q=0. \tag{2}$$

To verify this, we use ellipticity to see that $\exp t(P-P')\geqslant 0$ for all real t. Since (A,e) is an order unit space, it has the "nearest point property," i.e., $\forall a\in A$, $\exists b\in A^+$ such that $\|a-b\|=\operatorname{dist}(a,A^+)$. By [13, Theorem 1], this implies that

$$\langle (P-P')a,\rho\rangle=0 \qquad \text{whenever} \quad \langle a,\rho\rangle=0$$

for $a\in A^+$ and $\rho\in K$. In particular, for all $b\in A^+$ and $\rho\in K$, $\langle Qb,Q'^*\rho\rangle=0$ implies that

$$\langle (P-P')\,Qb,Q'^*\rho\rangle=0$$

from which (2) follows. Observe that an immediate consequence of (2) is

$$(Q-Q')(P-P')(Q-Q')=(Q+Q')(P-P')(Q+Q') \tag{3}$$

for all P-projections P, Q.

We next use (3) to establish

$$[P-P',Q-Q']^2e=0 \tag{4}$$

for all P-projections P, Q. In fact,

$$\begin{aligned}((P-P')&(Q-Q')-(Q-Q')(P-P'))^2e\\ &=(P-P')(Q-Q')(P-P')(Q-Q')e\\ &\quad+(Q-Q')(P-P')(Q-Q')(P-P')e\\ &\quad-(P-P')(Q+Q')(P-P')e\\ &\quad-(Q-Q')((P+P')(Q-Q')e.\end{aligned}$$

Substituting $e=(Q+Q')e$ and $e=(P+P')e$ in the last two terms on the right and using (3) now shows that (4) holds.

We now establish the last identity we need:

$$[P-P',Q-Q']e=0. \tag{5}$$

To verify this, note that $\exp t(P-P')\geqslant 0$ and $\exp t(Q-Q')\geqslant 0$ for all $t\in\mathbb{R}$ imply that $\exp t[P-P',Q-Q']\geqslant 0$ for all $t\in\mathbb{R}$ (cf. the argument in [12]). By (4)

$$0\leqslant \exp t[P-P',Q-Q']e=e+t[P-P',Q-Q']e$$

for all $t \in \mathbb{R}$; this implies (5). Finally, by [3, Corollary 3.7] A can be made into a JBW-algebra with normal state space K.

COROLLARY 1.6. *A compact convex set K is affinely isomorphic and homeomorphic to the state space of a* JB*-algebra (with w^*-topology) iff K is strongly spectral and elliptic.*

Proof. This follows at once from Theorem 1.5 and [3, Corollary 3.2, Proof of Corollary 3.8].

Remark 1.7. We have only used the consequence (1) of ellipticity for the proof of Theorem 1.5. Thus it follows from Lemma 1.4 that condition (1) is equivalent to ellipticity. In fact if condition (1) is satisfied for a spectral convex set K, then K is symmetric in the sense of [5], i.e., $2(P+P') - \mathbb{1} \geqslant 0$ for every P-projection P (cf. [5, Proposition 3.14]). This means with notations of Lemma 1.1 that the point $2\psi_F\rho - \rho$ obtained by reflecting ρ about the line segment $[\sigma, \tau]$ in the direction $\rho - \psi_F\rho$ is in K and K will be elliptic. This condition is formally very similar to Connes' [12] condition of facial homogeneity defined in the Hilbert space context. Thus the previous results connect the JBW theory developed in [2–7, 15] and [8–12]. Since this notion of ellipticity seems to be very important we give now equivalent definitions.

PROPOSITION 1.8. *Let K be a spectral convex set. Then the following are equivalent:*

(i) *K is elliptic.*

(ii) $\exp t(P - P') \geqslant 0$ *for all real t and all P-projections P.*

(iii) $|\langle (1 - (P+P'))a, \rho\rangle| \leqslant 2\langle Pa, \rho\rangle^{1/2}\langle P'a, \rho\rangle^{1/2}$ *for all a in A^+, all ρ in K, and all P-projections P.*

(iv) $P(Q - Q')P' = 0$ *for all P-projections P and Q.*

Proof. The equivalence between (i) and (ii) follows from the previous remark.

Assume (ii). The function $f\colon t \in \mathbb{R} \to \langle \exp t(P-P')a, \rho\rangle$ is positive and its minimum is $2\langle Pa, \rho\rangle^{1/2}\langle P'a, \rho\rangle^{1/2} + \langle (\mathbb{1} - (P+P'))a, \rho\rangle$. Now using the fact that K is symmetric (replace ρ by $(2(P+P') - \mathbb{1})^*\rho$) we obtain (iii).

Assume (iii). Thus for a in A^+, ρ in K, and Q a P-projection $(\langle Qa, \rho\rangle^{1/2} - \langle Q'a, \rho\rangle^{1/2}| \leqslant \langle a, \rho\rangle^{1/2}$. Replacing a by $P'a$ and ρ by $\|P\rho\|^{-1}P\rho$ for a P-projection P we obtain $\langle QP'a, P\rho\rangle^{1/2} - \langle Q'P'a, P\rho\rangle^{1/2} = 0$. This implies (iv).

Assume (iv). To prove (ii) we have by [13, Theorem 1] to show that if $\langle a, \rho\rangle = 0$ for a in A^+ and ρ in K, then $\langle (Q - Q')a, \rho\rangle = 0$ for all P-projections Q. In this case $\langle r(a), \rho\rangle = 0$ (cf. [2, Proposition 4.7]) and if P is the P-projection associated to $r(a)$ (i.e., $r(a) = Pe$), then $\langle P'e, \rho\rangle = \langle e - Pe, \rho\rangle = 1$ and $P'^*\rho = \rho$ by [2, Lemma 2.3]. Since $Pa = a$, we are done.

2. Characterization of Normal State Spaces of von Neumann Algebras

A. *Global* 3-*Ball*

Below A will denote a JBW-algebra with identity e and normal state space K. We begin by recalling some facts about the facial structure of K; see [5] for more background.

In [2] the notion of a *projective* fact of a convex set was defined. In our context, there is a 1–1 correspondence of idempotents p in A and projective faces of K, $p \leftrightarrow p^{-1}(1)$. Such faces occur in "quasi-complementary" pairs F and $F^{\#}$; if $F = p^{-1}(1)$, then $F^{\#} = (e-p)^{-1}(1) = p^{-1}(0)$.

For each projective face F of K, the convex set K is symmetric with respect to $\operatorname{co}(F \cup F^{\#})$ in the sense that there is a (unique) automorphism R_F of period two whose fixed point set is $\operatorname{co}(F \cup F^{\#})$. (For example, if $K = E^3$, the unit ball of $\mathbb{R}^3$, then each pair of antipodal boundary points are quasi-complementary projective faces and the associated reflection is just reflection with respect to the diameter connecting the points.) If $F = p^{-1}(1)$ and $s = 2p - e$, then R_F is just the dual map of conjugation by the symmetry $(a \to \{s, a, s\})$. We will call the reflections of K of the form R_F *projective reflections*.

Finally we remark that every norm-exposed of K is projective (A face is norm exposed if it is the intersection of K and a norm-closed supporting hyperplane). If A is the self-adjoint part of a von Neumann algebra, then every norm-closed face of K is projective.

Now if K is any convex set and if there exists a surjective affine map φ of K onto the standard 3-ball E^3, then we may think of K as being a "blown up" version of E^3, with each point σ in E^3 expanded to become $\varphi^{-1}(\sigma) \subseteq K$. To a minor extent, K will automatically possess some of the facial structure of E^3, e.g. the map $F \to \varphi^{-1}(F)$ sends faces of E^3 to faces of K and preserves intersections, but generally will not preserve l.u.b.'s in the lattice of faces. Below we give stronger conditions under which we say that φ gives K the structure of a "global 3-ball."

Definition 2.1. Let K be the normal state space of a JBW-algebra A. Then K is a *global* 3-*ball* if there exists a surjective affine map of K onto E^3 such that

(i) the map $F \to \varphi^{-1}(F)$ is an isomorphism of the lattice of faces of E^3 into that of K.

(ii) the map $R_F \to R_{\varphi^{-1}(F)}$ extends to an isomorphism of the group of affine automorphisms of E^3 into that of K.

Remarks 2.2. (i) Each face F of E^3 is norm exposed, and therefore so is $\varphi^{-1}(F)$; in particular each such face $\varphi^{-1}(F)$ is projective.

(ii) The first condition is equivalent to the formally weaker requirement that for antipodal points ρ, σ in $\partial_e E^3$ the smallest face of K containing their inverse images $\varphi^{-1}(\rho)$, $\varphi^{-1}(\sigma)$ is all of K. If this is true, then φ^{-1} preserves quasi-complements $\varphi^{-1}(F^{\#}) = \varphi^{-1}(F)^{\#}$ [2, Lemma 4.3], which immediately implies that φ^{-1} preserves l.u.b.'s.

(iii) The second condition could be weakened to the requirement that the map $R_F \to R_{\varphi^{-1}(F)}$ preserves products of reflections when defined, i.e., if $R_F R_G = R_H$, then $R_{\varphi^{-1}(F)} R_{\varphi^{-1}(G)} = R_{\varphi^{-1}(H)}$. This is in fact the only consequence which is used below.

(iv) Each face $F = p^{-1}(1)$ of K can be identified with the normal space of the JBW-algebra $U_p A$, and so we can ask whether F is a global 3-ball.

Lemma 2.3 is a global analogue of the fact that the state space of a JB-algebra A is isomorphic to E^3 iff A is isomorphic to $M_2(\mathbb{C})_{\text{sa}}$, the 2×2 self-adjoint matrices on the complex numbers.

LEMMA 2.3. *Let A be a* JBW*-algebra with normal state space K. Then K is a global* 3*-ball iff A is isomorphic to the self-adjoint part of $M_2(\mathbb{C}) \otimes B$ for a suitable von Neumann algebra B.*

Proof. Let $\varphi: K \to E^3$ be a surjective affine map satisfying properties (i) and (ii) of Definition 2.1. Identify E^3 with the state space of $M_2(\mathbb{C})_{\text{sa}}$, and let $\psi: M_2(\mathbb{C})_{\text{sa}} \to A$ be the map such that $\psi^* = \varphi$. As observed above, property (i) implies that φ^{-1} preserves quasi-complements, and so by [14, Proposition)1] ψ is a Jordan homomorphism such that $\psi(\mathbb{1}) = e$.

We are now going to construct symmetries s_1, s_2, s_3 in A such that

$$s_i \circ s_j = \delta_{ij} e, \tag{6}$$

and

$$U_{s_1} U_{s_2} U_{s_3} = id \tag{7}$$

where U_s denotes the triple product by s, i.e., $U_s a = \{s, a, s\}$. Let t_1, t_2, t_3 be symmetries in $M_2(\mathbb{C})_{\text{sa}}$ satisfying the relations corresponding to (6) and (7), e.g.,

$$t_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

and let G_1, G_2, G_3 be the corresponding faces in E^3, i.e., $G_i = q_i^{-1}(1)$, where $q_i = \frac{1}{2}(\mathbb{1} + t_i)$ for $i = 1, 2, 3$. Define $s_i = \psi(t_i)$. Then (6) follows from the fact that ψ is a Jordan homomorphism. Note that $U_{t_1} U_{t_2} U_{t_3} = id$, because $t_1 t_2 t_3 = -i\mathbb{1}$; dualizing gives $R_{G_3} R_{G_2} R_{G_1} = id$. If F_1, F_2, F_3 are the faces of K corresponding to s_1, s_2, s_3 (i.e., $F_i = p_i^{-1}(1)$, where $p_i = \frac{1}{2}(e + s_i)$), then $F_i = \varphi^{-1}(G_i)$. Now property (ii) implies that $R_{F_3} R_{F_2} R_{F_1} = id$, and by duality $U_{s_1} U_{s_2} U_{s_3} = id$, which establishes (7).

Now by the definition of the triple product, for $i \neq j$

$$\{s_i, s_j, s_i\} = 2s_i \circ (s_i \circ s_j) - s_i^2 \circ s_j = -s_j$$

and this implies $\{s_i, p_j, s_i\} = e - p_j$ for $i \neq j$. Thus p_j and $e - p_j$ are equivalent idempotents. It follows that A admits no homomorphisms onto M_3^8, since in M_3^8 the identity is not the sum of equivalent idempotents. Therefore, by [7, Theorem 9.5; 15, Corollary 2.4], A can be embedded as a weakly closed Jordan subalgebra of $B(H)_{sa}$, i.e., as a JW-algebra (see [18]). From now on we assume A is so embedded.

We next show that A is the self-adjoint part of the real norm-closed subalgebra $R(A)$ that it generates, or equivalently that A is reversible (see [17]). Let $\pi: A \to M$ be a Jordan homomorphism of A onto a σ-weakly dense subalgebra of a type I JBW-factor M with identity e_M. Let $w_i = \pi(s_i)$ for $i = 1, 2, 3$. Then $w_i \circ w_j = \delta_{ij} e_M$ and $U_{w_1} U_{w_2} U_{w_3} = id$. If w_4 were a fourth symmetry in M with $w_4 \circ w_i = \delta_{4i} e_M$, then $U_{w_i} w_4 = -w_4$ for $i = 1, 2, 3$ would contradict $U_{w_1} U_{w_2} U_{w_3} = id$. It follows that if M is a spin factor, 'then $\dim M = 4$ and M is reversible (in any representation) [6, Lemma 4.1]. Now the proof of [6, Lemma 4.5 and Theorem 4.6] shows the I_2 summand of A is reversible. By [16, Theorems 6.4 and 6.6] A is reversible.

Finally we show $R(A)$ can be given the structure of a C^*-algebra. First, define $j = s_1 s_2 s_3 \in R(A)$. Note that s_1, s_2, s_3 anti-commute by (6) and so $j^2 = -e, j^* = -j$. By (7)

$$s_1 s_2 s_3 a s_3 s_2 s_1 = a \qquad \text{for all} \quad a \in A,$$

and thus $ja(-j) = a$, i.e., $ja = aj$ for all $a \in A$, so j is central in $R(A)$. Since for $a \in R(A)$, $a = \frac{1}{2}(a + a^*) + \frac{1}{2}j(j(a^* - a))$, we have $R(A) = A \oplus jA$, and $R(A)$ can be viewed as a complex $*$-algebra with imaginary unit scalar j, and the involution inherited from $B(H)$. Observe that the norm inherited from $B(H)$ is a valid norm on $R(A)$ viewed as a complex linear space, since

$$\|(\alpha 1 + \beta j)a\|^2 = \|(\alpha 1 + \beta j)a)^*(\alpha 1 + \beta j)a\| = \|(\alpha^2 + \beta^2)\, a^* a\| = (\alpha^2 + \beta^2)\, \|a\|^2.$$

Then $R(A)$ becomes a C^*-algebra, with $A = R(A)_{sa}$. Since A has a predual, then $R(A)$ is von Neumann algebra in which the identity is the sum of equivalent projections (e.g., p_1 and $e - p_1$). Thus $R(A) \approx M_2(\mathbb{C}) \otimes B$, as claimed.

Conversely, let B be a von Neumann algebra, $A = (M_2(C) \otimes B)_{sa}$, and K the normal state space of A. We again identify E^3 with the state space of $M_2(\mathbb{C})$. Let $\psi: M(\mathbb{C})_{sa} \to A$ be the natural injection and let $\varphi = \psi^*$ so that φ is a surjection from K onto E^3. Since ψ is a unital Jordan homomorphism, then φ^{-1} preserves quasi-complements [14, Proposition 1] and as in Remark 2.2 K satisfies condition (i) of the definition of a global 3-ball. To verify (ii), note that $\alpha \to \alpha \otimes id$ is an isomorphism of the group of Jordan $*$-automorphisms of $M_2(\mathbb{C})$ into the corresponding group for $M_2(\mathbb{C}) \otimes B$. Dualizing gives an isomorphism of $\mathrm{Aut}(E^3)$ into $\mathrm{Aut}(K)$, and it is straightforward to verify that this sends R_F to $R_{\varphi^{-1}(F)}$ for each face F of E^3.

Remark 2.4. We note for later use that the proof to Lemma 2.3 has shown that if A is represented as a JW-algebra, then $R(A)$ is isomorphic (as real $*$-algebra) to $M_2(\mathbb{C}) \otimes B$ for a von Neumann algebra B.

B. *The Global* 3-*ball Property*

DEFINITION 2.5. Let K be the normal state space of a JBW-algebra A. Faces F and G of K are *equivalent* if there exists a finite sequence $R_1, R_2, \ldots, R_m$ of projective reflections whose composition maps F onto G.

Remark 2.6. (i) If K is the state of a C^*-algebra (viewed as the normal state space of the enveloping von Neumann algebra), then pure states σ and τ are unitarily equivalent iff the faces $\{\sigma\}$, $\{\tau\}$ are equivalent in the sense above.

(ii) Let p and q be idempotents in A. Then p and q are said to be equivalent if there is a sequence of symmetries $s_1, \ldots, s_n$ such that the composition of the maps $U_{s_1}, \ldots, U_{s_n}$ maps p onto q, cf. [18, 7]. Note that p and q will be equivalent iff the faces $p^{-1}(1)$ and $q^{-1}(1)$ are equivalent.

DEFINITION 2.7. Let K be the normal state space of a JBW-algebra A. Then K has the *global* 3-*ball property* if for each pair F, G of orthogonal, equivalent faces of K, the face $F \vee G$ is a global 3-ball (cf. Remark 2.2(iv)).

Remark 2.8. (i) It follows from Theorem 2.9 and [18, Theorem 6] that "orthogonal" above could be replaced by "disjoint."

(ii) Let A be a JB-algebra with state space K (viewed as a normal state space of the JBW-algebra B^{**}. If $F = \{\sigma\}$ and $G = \{\tau\}$ (i.e., σ, τ are extreme points in K), then $\{\sigma\} \vee \{\tau\}$ will be a Hilbert ball denoted by $B(\sigma, \tau)$ [5, Theorem 3.11]. The faces F and G are equivalent iff $\dim B(\sigma, \tau) \geqslant 2$ [6, Proposition 2.3]. It is possible to choose equivalent orthogonal pure states σ' and τ' such that $B(\sigma', \tau') = B(\sigma, \tau)$. Thus if K has the global 3-ball property, $B(\sigma, \tau)$ must be a global 3-ball. In fact this holds iff $B(\sigma, \tau) = E^3$, since the only spin factor isomorphic to the self-adjoint part of a von Neumann algebra is the 4-dimensional one. Note as a consequence that the global 3-ball property for JB-algebras implies the 3-ball property.

THEOREM 2.9. *Let A be a* JBW*-algebra with normal state space K. Then A is isomorphic to the self-adjoint part of a von Neumann algebra iff K has the global* 3*-ball property.*

Proof. Assume first that A is the self-adjoint part of a von Neumann algebra $\mathfrak{A}$. If F and G are equivalent orthogonal faces of K, then the corresponding projections p and q are equivalent and orthogonal. It follows that $(p+q)\mathfrak{A}(p+q)$ contains a set of 2×2 matrix units, and so $\mathfrak{A} \approx M_2(\mathbb{C}) \otimes B$ for some von Neumann algebra B. Now by Lemma 2.3, the normal state space can be identified with $(p+q)^{-1}(1) = F \vee G$, and we have shown K has the global 3-ball property.

Conversely, assume K has the global 3-ball property. We first show that A can be embedded as a weakly closed Jordan subalgebra of $B(H)_{\mathrm{sa}}$. By [7, Theorem 9.5; 15, Corollary 2.4], it suffices to show that A has no nontrivial homomorphism onto M_3^8. Suppose $\pi: A \to M_3^8$ were such a homomorphism. By [15, Lemma 3.6] we could choose equivalent idempotents p and q in A mapping onto the matrix units e_{11} and e_{22} in M_3^8. Then π would map $\{(p+q, A, (p+q)\}$ onto the 10-dimensional spin factor $\{(e_{11}+e_{22}), M_3^8, (e_{11}+e_{22})\}$. But by Lemma 2.3, $\{(p+q), A, (p+q)\}$ is isomorphic to the self-adjoint part of a von Neumann algebra, and so by [6, Lemma 3.4] cannot be mapped onto the 10-dimensional spin factor.

Thus we assume A is a weakly closed Jordan subalgebra of $B(H)_{\mathrm{sa}}$. If the identity of A is the sum of two equivalent projections, by Lemma 2.3 we are done. Then by the structure theory for JW-algebras, we may assume that A is of type I, and even is homogeneous, i.e., the identity e of A is the σ-weak sum of equivalent Abelian projections [18, Theorems 16 and 17]. Grouping these projections, we can assume for the same reason $e = e_1 + \cdots + e_n$, where $3 \leqslant n < \infty$ and n is odd. In particular, A is reversible by [16, Lemma 3.1] and $A = R(A)_{\mathrm{sa}}$.

Now choose unitarities $u_1, \ldots, u_n \in R(A)$ (products of symmetries) such that $u_j e_1 u_j = e_j$ for $1 \leqslant j \leqslant n$. Define

$$e_{ij} = e_i u_i e_1 u_j^* e_j.$$

Then $\{e_{ij}\}$ are matrix units for $R(A)$, so $R(A) \approx M_n(B)$ for some real $*$-algebra B.

Now by Remark 2.4, $R((e_{11}+e_{22})A(e_{11}+e_{22}) \subseteq (e_{11}+e_{22})R(A)(e_{11}+e_{22})$ is isomorphic (as a real $*$-algebra) to a von Neumann algebra, and so contains a central element j_{12} such that $j_{12}^2 = -(e_{11}+e_{22})$, $j_{12}^* = -j_{12}$. Let $j_1 = e_{11} j_{12}$, then $j_1^2 = -e_{11}$, $j_1^* = -j_1$, and j_1 is central in $e_{11} R(A) e_{11} \approx B$. If j' is the corresponding element of B, then $j = \sum_i j' e_{ii}$ is central in $M_n(B)$,

$j^2 = -e$, $j^* = -j$. As in the proof of Lemma 2.3, we conclude that $R(A) \approx M_n(B)$ is $*$-isomorphic to a C^*-algebra, and so $A = R(A)_{sa}$ is isomorphic to the self-adjoint part of a C^*-algebra. Since A is a dual space, then A is isomorphic to the self-adjoint part of a von Neumann algebra. ∎

We can now characterize normal state spaces of von Neumann algebras:

THEOREM 2.10. *A convex set K is affinely isomorphic to the normal state space of a von Neumann algebra iff K is spectral, elliptic, and has the global 3-ball property.*

REFERENCES

1. E. M. ALFSEN, Compact convex sets and boundary integrals, "Ergebnisse der Math.," Vol 57, Springer-Verlag, Berlin, 1971.
2. E. M. ALFSEN AND F. W. SHULTZ, Non-commutative spectral theory for affine function spaces on convex sets, *Mem. Amer. Math. Soc.* **172** (1976).
3. E. M. ALFSEN AND F. W. SHULTZ, On non-commutative spectral theory and Jordan algebras, *Proc. London Math. Soc.* (3) **38** (1979), 497–516.
4. E. M. ALFSEN AND F. W. SHULTZ, "State Spaces of Jordan Algebras," Oslo preprint, 1976.
5. E. M. ALFSEN AND F. W. SHULTZ, State spaces of Jordan algebras, *Acta Math.* **140** (1978), 155–190.
6. E. M. ALFSEN, H. HANCHE-OLSEN, AND F. W. SHULTZ, State spaces of C^*-algebras, *Acta Math.* **144** (1980), 267–305.
7. E. M. ALFSEN, F. W. SHULTZ, AND E. STØRMER, A Gelfand–Neumark theorem for Jordan algebras, *Advan. Math.* **28** (1978), 11–56.
8. J. BELLISSARD, B. IOCHUM, AND R. LIMA, Homogeneous and facially homogeneous self-dual cones, *Linear Algebra and Appl.* **19** (1978), 1–16.
9. J. BELLISSARD AND B. IOCHUM, Homogeneous self-dual cones versus Jordan algebras: The theory revisited, *Ann. Inst. Fourier* (*Grenoble*) **18** (1978), 27–67.
10. J. BELLISSARD AND B. IOCHUM, Spectral theory for facially homogeneous symmetric self-dual cones, *Math. Scand.* **45** (1979), 118–126.
11. J. BELLISSARD AND B. IOCHUM, L'algèbre de Jordan d'un cône autopolaire facialement homogène, *C. R. Acad. Sci. Paris, Sér. A* **288** (1979), 229–232.
12. A. CONNES, Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann, *Ann Inst. Fourier* (*Grenoble*) **24** (1974), 121–155.
13. D. E. EVANS AND H. HANCHE-OLSEN, The generators of positive semigroups, *J. Funct. Anal.* **32** (1979), 207–212.
14. F. W. SHULTZ, Dual maps of Jordan homomorphisms and $*$-homomorphisms between C^*-algebras, *Pacific J. Math.* **93** (1981), 435–441.
15. F. W. SHULTZ, On normed Jordan algebras which are Banach dual spaces, *J. Funct. Anal.* **31** (1979), 360–376.
16. E. STØRMER, Jordan algebra of type I, *Acta Math.* **115** (1966), 165–184.
17. E. STØRMER, On the Jordan structure of C^*-algebras, *Trans. Amer. Math. Soc.* **120** (1965), 438–447.
18. D. TOPPING, Jordan algebras of self-adjoint operators, *Mem. Amer. Math. Soc.* **53** (1965).